FINITELY GENERATED SEMIGROUPS WITH SUCH A PRESENTATION THAT ALL THE CONGRUENCE CLASSES ARE CONTEXT-FREE LANGUAGES (Languages, Computations, and Algorithms in Algebraic Systems)

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FINITELY GENERATED SEMIGROUPS WITH SUCH A PRESENTATION THAT ALL THE CONGRUENCE CLASSES ARE CONTEXT-FREE LANGUAGES*

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Abstract In this paper, we investigate finitely generated semigroups with such a presentation that all the congruence classes are context-free languages.

A monoid M is called *finitely generated* if there exists a finite set of X and there exists a surjective homomorphism of X^* to M which maps an empty word onto the identity element of M.

1. Presentations of monoids

Definition 1 . (1) *Let X be finite alphabets and R a subset of $X^* \times X^*$. Then R is string-rewriting system.*
(2) *For $u, v \in X^*$, $(w_1, w_2) \in R$, $uw_1v \Rightarrow_R uw_2v$.*
The congruence μ_R on X^* generated by $\Rightarrow_R$ is the Thue congruence defined by R.
(3) *A monoid M is (finitely)presented if there exists a (finite) set of X, there exists a surjective homomorphism ϕ of X^* to S and there exists a (finie) string-rewriting system R consisting of pairs of words over X such that the Thue congruence μ_R is the congruence $\{(w_1, w_2) \in X^* \times X^* \mid \phi(w_1) = \phi(w_2)\}$.*

Definition 2 . *A monoid M has a presentation with [finite, regular, contex-free] congruence classes if there exists a finite set X and there exists a surjective homomorphism ϕ of X^+ to M such that for each words $w \in X^+$, $\phi^{-1}(\phi(w))$ is a [finite, regular, contex-free] language.*

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*This is an absrtact and the paper will appear elsewhere.

2. Syntactic monoids of languages and finitely generated presented monoids

Definition 3 . *Let A be finite alphabets and A^* the set of words over A. A subset L of A^* is called a language. The syntactic congruence σ_L on A^* is defined by $w\sigma_L w'$ $(w, w' \in A^*)$ if and only if $\{(x,y) \in A^* \times A^* \mid xwy \in L\} = \{(x,y) \in A^* \times A^* \mid xw'y \in L\}$.*
Then a factor monoid A^*/σ_L is called the syntactic monoid of L.

Example 1 . *$A = \{a_1, \cdots, a_n\}$. For any $w = b_1 b_2 \cdots b_r$, let $w^R = b_r \cdots b_2 b_1$. Let $L = \{ww^R | w \in A^*\}$. Then*
(1) L is a context-free language which is not accepted by any deterministic pushdown automata.
(2) $Syn(L)$ is the free monoid A^* on A.
That is, $\phi : A^* \to Syn(L)(w \mapsto \sigma_L w)$ is an isomorphism.

Definition 4 . *Let M be a monoid and m an element of M. The syntactic congruence σ_m on M is defined by $s\sigma_m t$ $(s, t \in M)$ if and only if $\{(x,y) \in M \times M \mid xsy = m\} = \{(x,y) \in M \times M \mid xty = m\}$.*
The factor monoid M/σ_m is called the syntactic monoid of M at m.

Lemma 1. *Let L be a language of X^*. Then L is a union of σ_L-classes in X^*.*

Proposition 1. *Let L be a language of A^* and L^c the complement of the set L in A^*. Then $Syn(L) = Syn(L^c)$.*

Theorem 1 . *Let L be a language of X^*. Then the following are equivalent :*
(1) L is a σ_L-class in X^*.
(2) $xLy \cap L \neq \emptyset$ $((x, y \in X^*) \Rightarrow xLy \subseteq L$.
(3) L is an inverse image $\phi^{-1}(m)$ of a homomorphism ϕ of X^* to a monoid M.

Theorem 2 . (*Shoji [S]*) *Let M be a finitely generated monoid and ϕ a surjective homomorphism of A^* to M. For m an element of M, let $L = \phi^{-1}(m)$.*
Then the syntactic monoid $Syn(L) = A^*/\sigma_L$ of L is isomorphic to the syntactic monoid M/σ_m of M at m.

3. Finitely generated semigroups with such a presentation that all the congruence classes are context-free languages

Theorem 3 . (*Shoji* [*S*]) *A finitely generated semigroup* S *has a presentation with regular congruence classes if and only if for any* $s \in S$, S/σ_s *is a finite semigroup.*

Theorem 4 . (*Shoji* [*S*]) *Let* S *be a finitely generated semigroup.*
Then S *has a presentation with finite congruence classes if and only if the following are satisfied :*

(1) S *has no idempotent.*

(2) *For any* $s \in S$, S/σ_s *is a finite nilpotent semigroup with a zero element* 0.

Example 2 . *Let* $A = \{a, b\}$ *and a context-free language* $L = \{a^n b^n, b^n a^n | n \in \mathbb{N}\}$. *Then all of* σ_L*-classes are* $\{1\}$, $\{ab\}$, $\{a^n\}$, $\{b^n\}$, $c_n = \{a^{p+n} b^p | p \in \mathbb{N}\}$, $d_n = \{a^q b^{q+n} | q \in \mathbb{N}\}$, $\{ba\}$, $e_n = \{b^p a^{p+n} | p \in \mathbb{N}\}$, $f_n = \{b^{q+n} a^q | q \in \mathbb{N}\}$. *Hence* $Syn(L)$ *has a regular cross-section. Also,* $Syn(L) - \{1\}$ *is a* $\mathcal{D}$*-class.* $Syn(L)$ *has a representation with context-free congruence classes.*

Example 3 . *Let* $A = \{a, b\}$ *and* $G : S \to SSS|aSb|\epsilon$. *Then* G *is a context-free grammar and its accepted language* $L(G)$ *equals to* $\{a^n b^n | n \geq 0\}$.
The syntactic monoid $Syn(L(G))$ *has the presentation* $A^*/\{ab = 1\}$. *It is easily seen that* $Syn(L(G))$ *has a representation with context-free congruence classes.*

Example 4 . *Let* $A = \{a_1, \cdots, a_r\} \cup \{b_1, \cdots, b_r\}$ *and* $F(A)$ *the free inverse semigroup over* A. *Then there exists the canonical map* $\phi : A^* \to F(A)$ $(b_i \mapsto a_i^{-1})$ *such that for each* $w \in F(A)$, $\phi^{-1}(w)$ *is not a context-free language. Thus, Free inverse semigroups do not have a representation with context-free congruence classes.*

Remark. Even a monogenic free inverse smigroup do not have any representation with context-free congruence classes.

Result 1 . *For every finitely generated group* G, *there exists a language* L *of* A^* *such that* G *is isomorphic to* $Syn(L)$.

Result 2 . (*Muller and Schupp* [*MS*]) (1) *Every finitely generated vertually free group* G *has a (monoid)-representation with context-free congruence classes.*
(2) *Conversely, if a finitely generated group* G *has a (monoid)- representation with context-free congruence classes then* G *is a vertually free group.*

Theorem 5 . *Let S be a semigroup having a representation with context-free congruence classes. If S is a completely $(0-)$ simple semigroup, then both the $\mathcal{L}$-classes and the $\mathcal{R}$-classes of S is finite and the maximal subgroup is vertually free.*

Theorem 6 . *Let S be a finitely generated submonoids of a vertually free group G. Then S is a cancellative monoid having a representation with context-free congruence classes.*

Example 5 . *Let A be finite alphabets containing $\{a, b, c\}$. Let $R = \{(acb, c)\}$ be a string-rewriting system on A^*. The monoid $M = A^*/\mu_R$ has a represntation with context-free congruence classes. Moreover, M is a cancellative monoid which is embedded in a group $G = < a, b, c \mid c^{-1}ac = b^{-1} >$ which is not vertually context-free.*

Theorem 7 . *Let M_1, M_2 be a finitely generated monoids having a presentation with context-free congruence classes. Then the free product $M_1 * M_2$ of M_1, M_2 has a presentation with context-free congruence classes.*

References

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